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Small Displacements in Parabolic Reflectors*

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ABSTRACT

The effect of small structural displacements in parabolic reflectors are analyzed. Axial and lateral feed displacements and in Cassegrain systems similar displacements and rotation of the secondary are considered. The resultant aperture phase error, loss of gain and beam displacement are presented.

I. Introduction

It is frequently required to set up structural specifications for large parabolic reflectors. The tolerance of the surface, of the feed focus and of the Cassegrain subreflector, if any, are important parameters that markedly affect the cost of the structure. For an intelligent decision the effect of these deviations on the antenna gain, on the boresight shift and on the radiation pattern must be obtained. In general, this can be done by the evaluation of the radiation integral with and without the mechanical distortions.[1] This, however, is a complex procedure especially for non-axially symmetric distortions and requires computer programming. It is the purpose of this paper to determine the aperture phase distribution due to the mentioned distortions when they are very small compared to the dimensions of the system. This knowledge permits us to use existing literature[2,3] to estimate the pattern distortion and in addition when the phase deviations are small in radian measure to determine the loss of gain.

The subject of antenna surface tolerance has been considered elsewhere[4] and will not be repeated here, except to note for completeness that the gain loss due to a surface rms error "ϵ" (one-half the rms path length) is:

$$\frac{G}{G_0} = e^{-\left(\frac{4\pi\epsilon}{\lambda}\right)^2} \qquad (1).$$

This simple formula holds for any smooth aperture illumination and is valid for large gain losses (several db)[5] provided the surface errors are uniformly distributed over the aperture.

II. Analysis

We wish to determine the aperture phase front deviation when the feed is displaced either axially or laterally and in Cassegrain systems when the subreflector is similarly displaced or rotated about its vertex. This is done by determining the path length error of a general ray from the feed to the aperture plane by geometric optics considering the angle of incidence and the slope of the reflecting surface. The mathematics is tedious, but straight forward with surprisingly simple final results (see Appendix for example). They are given in Table I for various system displacements. The main reflector or subreflector rotations are taken about the reflector vertices. The Appendix also defines the geometry and summarizes the relations of the various surfaces permitting the path length error to be expressed in terms of the aperture radial coordinates. For the Cassegrain geometries the reader will find it instructive to consider the values in the table for the limiting cases of unity and very high magnifications where the results are physically obvious. ?

The gain of a circular aperture with an arbitrary phase error $\delta(r, \phi)$ may be written as

$$G = \frac{4\pi}{\lambda^2} \frac{\left| \int_0^{2\pi} \int_0^1 f(r, \phi)\, e^{j\delta(r,\phi)}\, r\, dr\, d\phi \right|^2}{\int_0^{2\pi} \int_0^1 f^2(r, \phi)\, r\, dr\, d\phi} \qquad (2)$$

where $f(r, \phi)$ is the aperture illumination function.

TABLE 1

PATH LENGTH ERRORS

<table>
<tr><th></th><th>Focal Fed</th><th>Cassegrain</th><th>GREGORIAN</th></tr>
<tr><td>~~Feed~~ Displacement
AXIAL – ΔY</td><td>$\Delta Y(1-\cos\theta_p)$</td><td colspan="2">$\Delta Y(1-\cos\theta_f)$</td></tr>
<tr><td>Feed Displacement
ATERAL – ΔX</td><td>$\Delta X \sin\theta_p \cos\phi$</td><td colspan="2">$\Delta X \sin\theta_f \cos\phi$</td></tr>
<tr><td>MAIN REFLECTOR
TATION – $\Delta\alpha$</td><td>$f\Delta\alpha\left[\frac{r}{f}+\sin\theta_p\right]\cos\phi$</td><td colspan="2">$f\Delta\alpha\left[\frac{r}{f}+\sin\theta_p\right]\cos\phi$</td></tr>
<tr><td>SUB Reflector
XIAL – ΔY</td><td>———</td><td colspan="2">$\Delta Y\left[(1-\cos\theta_p)+(1-\cos\theta_f)\right]$</td></tr>
<tr><td>SUB Reflector
ATERAL – ΔX</td><td>———</td><td>$\Delta X\left[\sin\theta_p-\sin\theta_f\right]\cos\phi$</td><td>$\Delta X\left[\sin\theta_p+\sin\theta_f\right]\cos\phi$</td></tr>
<tr><td>ub-Reflector
TATION – $\Delta\alpha$</td><td>———</td><td colspan="2">$\Delta\alpha\lvert c-a\rvert\left[\sin\theta_p+M\sin\theta_f\right]\cos\phi$</td></tr>
</table>

12 DB TAPER

EDGE / ∠ ERROR	LOSS DB
45°	0.2 DB
90°	.8 DB
180°	3.5 DB
360°	13 DB

For small phase errors the exponential may be expanded in a power series with the result that the ratio of the gain to the no-error gain G_o is:[4]

$$\frac{G}{G_o} \cong 1 - \overline{\delta^2} + \overline{\delta}^2 \tag{3}$$

where

normalized

$$\overline{\delta^2} = \frac{\int_0^{2\pi}\int_0^1 f(r,\phi)\,\delta^2(r,\phi)\,r\,dr\,d\phi}{\int_0^{2\pi}\int_0^1 f(r,\phi)\,r\,dr\,d\phi}$$

$$\overline{\delta} = \frac{\int_0^{2\pi}\int_0^1 f(r,\phi)\,\delta(r,\phi)\,r\,dr\,d\phi}{\int_0^{2\pi}\int_0^1 f(r,\phi)\,r\,dr\,d\phi}$$

Axial Loss due to Axial Displacement (ALAD)

Axial displacements of the feed in focal fed or Cassegrain systems lead to phase errors of the type:

un-normalized

$$\delta = \left(\frac{2\pi\Delta}{\lambda}\right)(1-\cos\theta) = \left(\frac{2\pi\Delta}{\lambda}\right)\frac{2\left(r/2f\right)^2}{1+\left(r/2f\right)^2} \tag{4}$$

The phase error is axially symmetric and consists of second and higher even power terms. For an illumination taper of $f(r) = 1 - ar^2$ the loss of gain (3) can be evaluated in closed form and written as:

$$\frac{G}{G_0} = 1 - \frac{\left(\frac{2\pi\Delta}{\lambda}\right)^2}{3\left(\frac{4f}{D}\right)^4} \text{ALAD} \qquad (5)$$

where ALAD is a correction factor depending on the illumination taper and the focal length. ALAD approaches unity in the limit of long focal length and uniform illumination. It is given in Fig. 1. It should be noted that the gain loss is proportional to the square of the axial displacement and the inverse fourth power of the effective focal length or in Cassegrain systems to the inverse fourth power of the magnification.

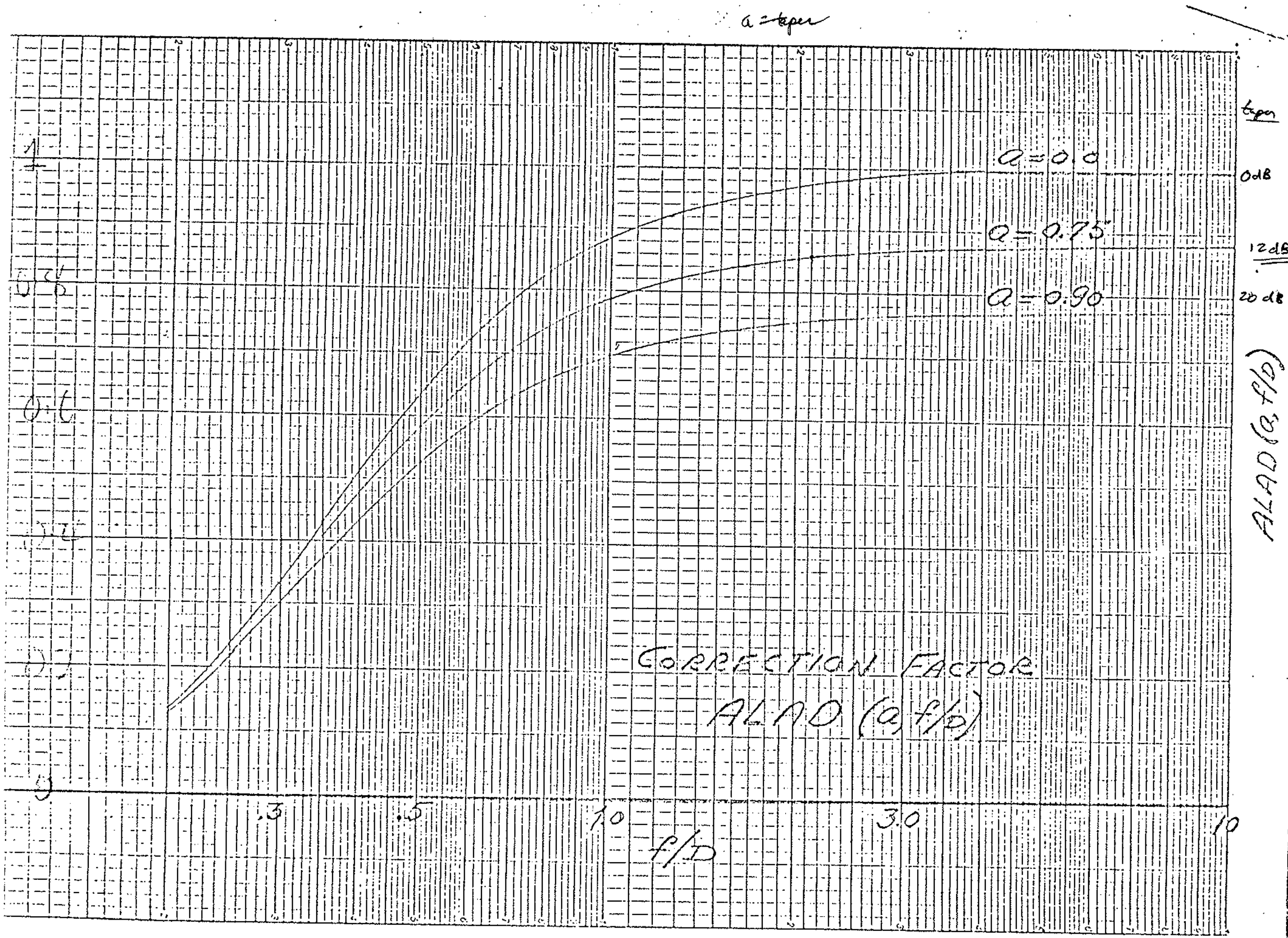
For the more complex axial displacement of the Cassegrain subreflector, it is necessary to evaluate (3) with the table indicated phase error. This may be done by numerical integration; however, as a practical matter, a very accurate result is obtained by using a pure square law phase error with the same rim value; that is using the value ALAD (a, ∞).

Axial Loss due to Lateral Displacement (ALLD)

Lateral displacement of the feed causes a boresight beam shift with a consequent loss of axial gain.

The aperture phase errors are of the type:

$$\delta = \left(\frac{2\pi\Delta}{\lambda}\right) \sin\theta \cos\phi = \left(\frac{2\pi\Delta}{\lambda}\right) \frac{r/f}{1+\left(r/2f\right)^2} \cos\phi \qquad (6)$$


a = taper
taper
a = 0.0
0 dB
a = 0.75
12 dB
a = 0.90
20 dB
ALAD (a, f/D)
1
0.8
0.6
0.4
0.2
0
.3
.5
1.0
3.0
10
f/D
CORRECTION FACTOR
ALAD (a f/D)

and the axial loss is:

$$\frac{G}{G_o} = 1 - \frac{2\left(\frac{2\pi\Delta}{\lambda}\right)^2}{\left(\frac{4f}{D}\right)^2} \text{ ALLD} \tag{7}$$

where the correction factor ALLD is given in Fig. 2.

For the more complex subreflector lateral displacement or reflector rotations the axial loss must be determined from (3) by numerical integration using the tabulated values.

Beam Peak Loss due to Lateral Displacement (BPLLD)

Frequently the specifications on the lateral displacements or subreflector rotations can be relaxed by repositioning the entire antenna to the new beam peak position. This occurs in radio astronomy, and in many radio relay and radar applications.

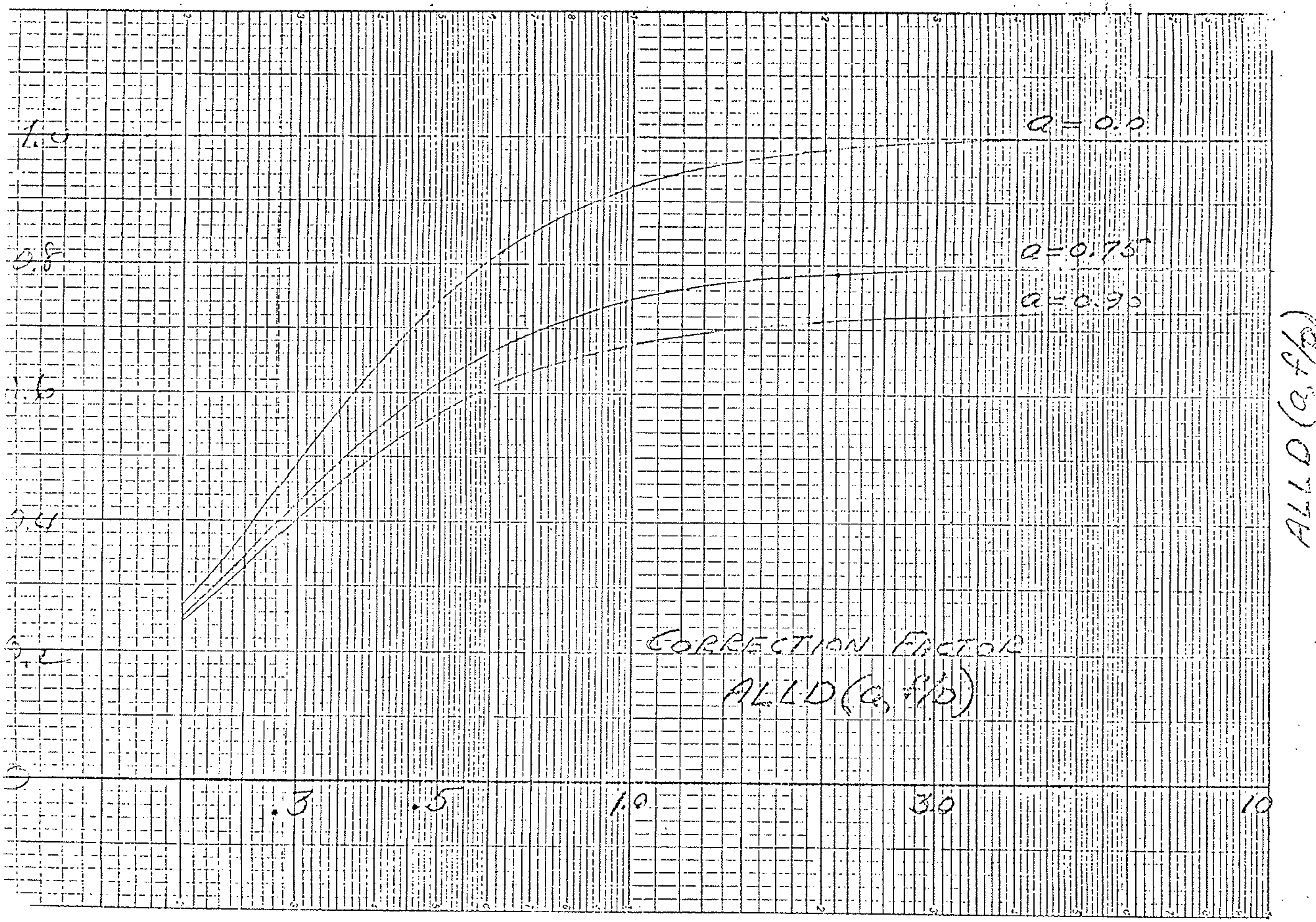
To find the beam peak we express the aperture phase error with respect to a plane inclined at an arbitrary angle "θ'" to the aperture. Insertion in (3) and then differentiation with respect to "θ'" to find the maximum gain will obtain the boresight angle "θ'_m".

The phase error may be written as:

$$\delta = \frac{2\pi}{\lambda}\left[r \sin\theta' \cos\phi - \sum_n \Delta_n \sin\theta_n \cos\phi\right]$$

where for generality the summation includes several lateral displacement terms. As $\bar{\delta} = 0$ and letting $u = \sin\theta'$ we have

NO. 32,238-R. 10 DIVISIONS PER INCH (70 DIVISIONS) BY TWO 3-INCH CYCLES RATIO RULING. PRINTED IN U.S.A.



$$\frac{\partial}{\partial u}\int_0^{2\pi}\int_0^1 f(r)\left[u-\sum_n \frac{\Delta_n}{r}\sin\theta_n\right]^2 \cos^2\phi \, r^3 \, dr \, d\phi = 0$$

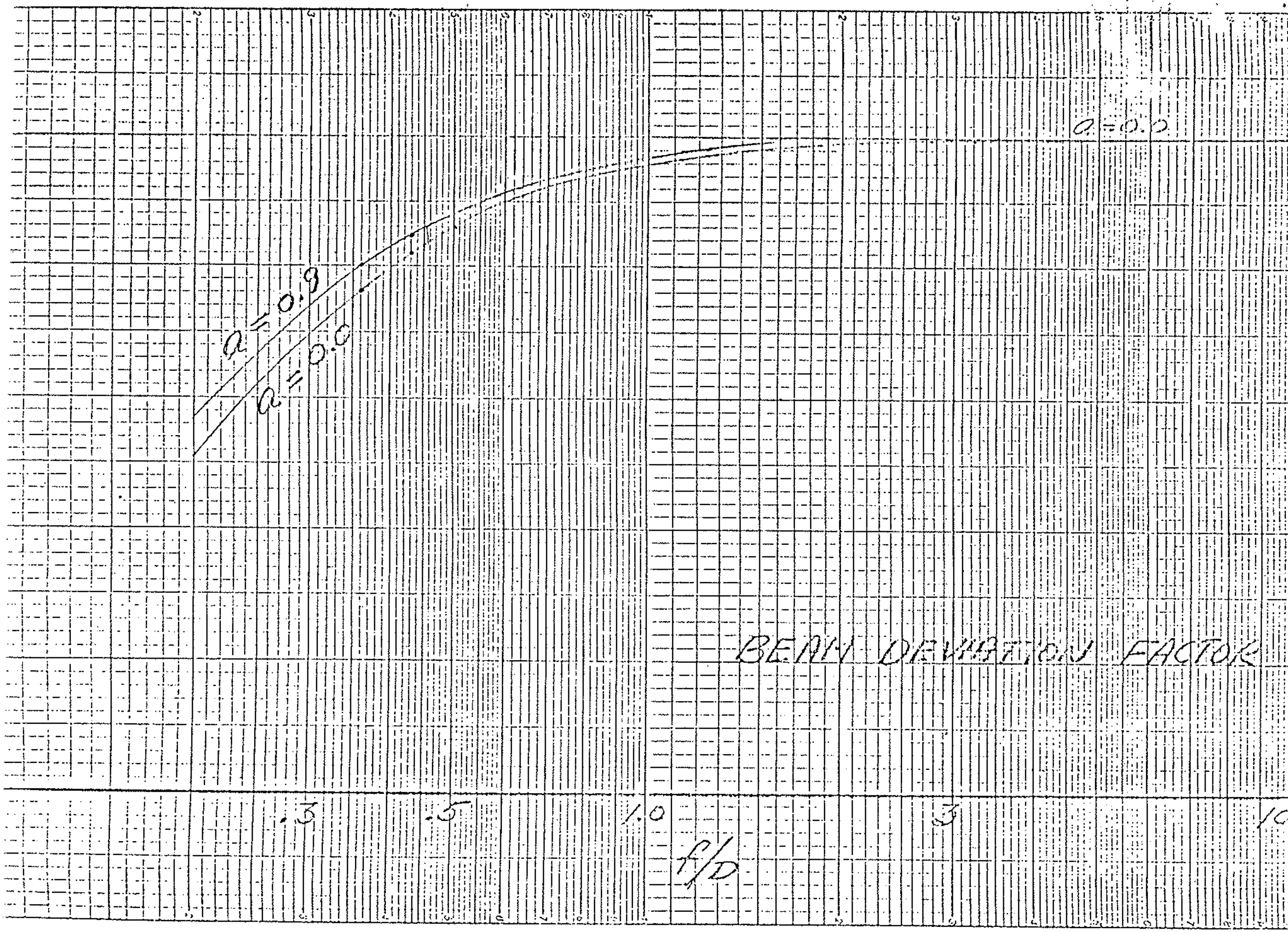
with the result

$$u_m = \sum_n \frac{\Delta_n}{f_n} \frac{\int_0^1 \frac{f(r)\, r^3 dr}{1+(r/2f)^2}}{\int_0^1 f(r)\, r^3 dr} = \sum_n \frac{\Delta_n}{f_n} BDf_n \qquad (8)$$

where BDf_n is the beam deviation factor[3,6]

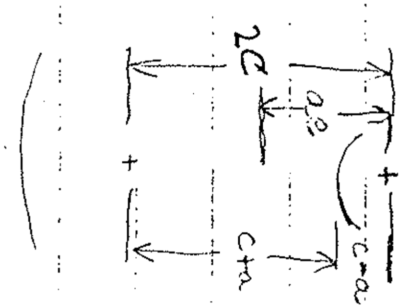
$$BDf_n = \frac{\int_0^1 \frac{f(r)\, r^3 dr}{1+(r/2f)^2}}{\int_0^1 f(r)\, r^3 dr}$$

The BDf_n depends on the aperture taper and the focal length and is given in Fig. 3 for our parabolic illumination. Equation (8) indicates that the beam displacement is the sum of the component terms. On this basis we can prepare Table II giving the beam displacements for the various reflector and feed displacements.



BEAM DEVIATION FACTOR
a = 0.9
a = 0.0
a = 0.0
BDF
1.0
9.0
8.0
7.0
6.0
f/D
.3
.5
1.0
3
10

	PARABOLIC	CASSEGRAIN	GREGORIAN
ed Displacement LATERAL - ΔX	$\frac{\Delta X}{f} BDf$	$\frac{\Delta X}{F} BDF$	
ain Reflector Rotation - $\Delta\alpha$	$\Delta\alpha [1 + BDf]$	$\Delta\alpha [1 + BDf]$	
ub-Reflector TERAL - ΔX	———	$\overset{\sim \Delta x / f}{\frac{\Delta X}{f} BDf} - \frac{\Delta X}{F} BDF$	$\frac{\Delta X}{f} BDf + \frac{\Delta X}{F} BDF$
b- Reflector TATION - $\Delta\alpha$	———	$\Delta\alpha \frac{\lvert c-a \rvert}{f} [BDf + BDF] \sim \frac{2\Delta\alpha}{m}$	



$c - a = \frac{c+a}{m} \sim \frac{f}{m}$

$\sim \frac{\Delta x}{f}$

TABLE II

Knowing the beam position we can determine the gain loss from (3) as our phase error for a lateral feed displacement is:

$$\delta = \left(\frac{2\pi\Delta}{\lambda}\right)\left[BDf - \frac{1}{1+(r/2f)^2}\right]\frac{r}{f}\cos\phi \qquad (9)$$

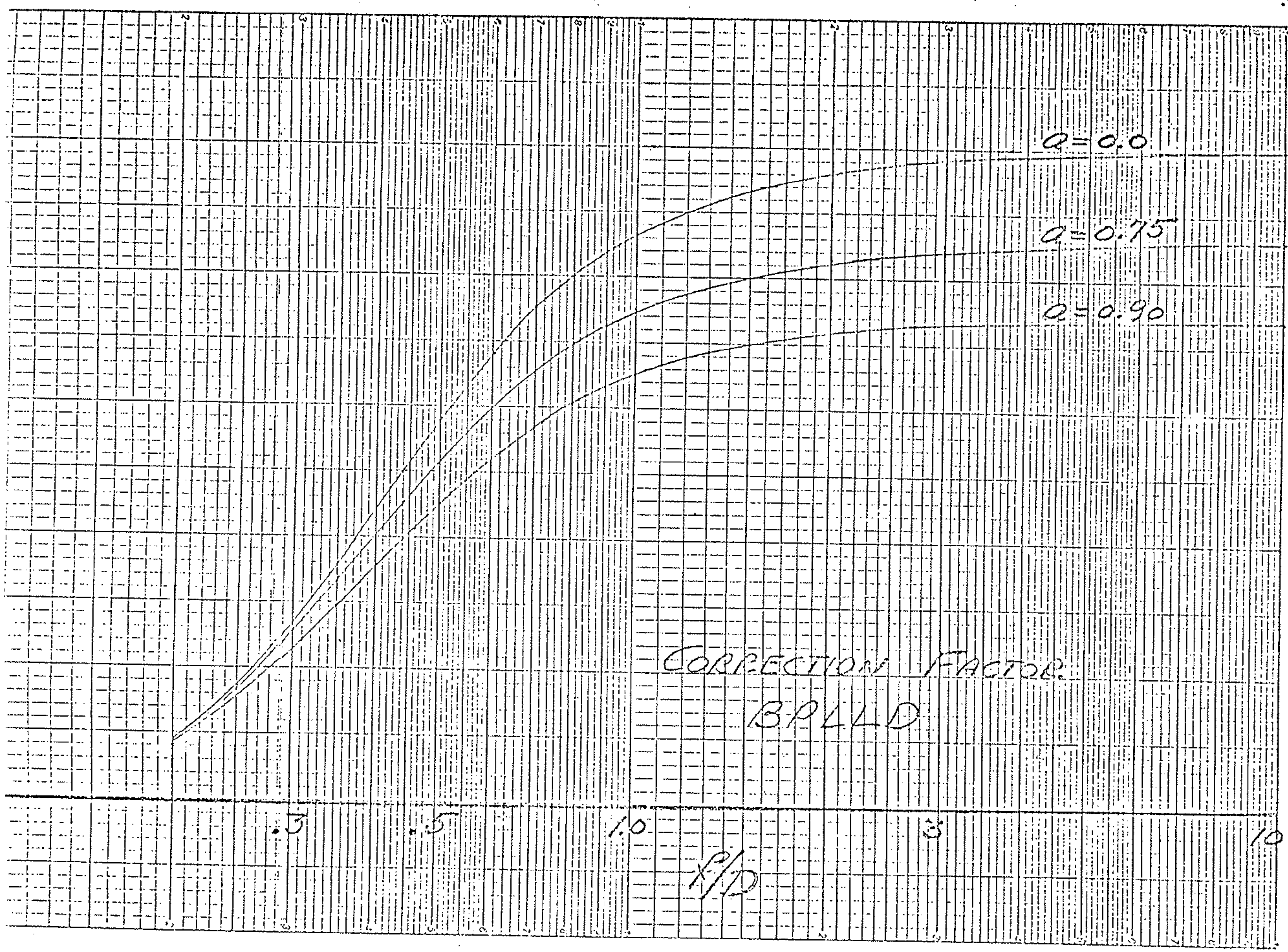
For the parabolic illumination the loss of gain at beam peak may be evaluated in closed form and written:

$$\frac{G}{G_0} = 1 - \frac{\left(\frac{2\pi\Delta}{\lambda}\right)^2}{18\left(\frac{4f}{D}\right)^6} BPLLD$$

where the correction factor BPLLD is given in Fig. 4. Again for the more complex subreflector displacements the gain loss must be determined by numerical integration of (3).

III. Discussion

In any paper using an approximation technique it is necessary to discuss the accuracy of the results. As structural strains are exceedingly small compared to the aperture dimensions, the expansions leading to the phase errors are certainly permitted. The results as given in Table I then may be used with confidence in determining the radiation characteristics (pattern and gain) by the use of the radiation integral. However, the use of the exponential expansion in Equation (2) is much more restrictive. As


a=0.0
a=0.75
a=0.90
CORRECTION FACTOR
BPLLD
.3
.5
1.0
3
10
r/D
BPLLD (a,r/D)

we use only the first three terms, we would expect our further analysis to be valid only for aperture rim phase errors of less than a radian. Fortunately, comparison of these results with the exact solution[7] for a uniform aperture with a quadratic phase error (large f/D) indicates agreement within 0.1 db for a rim phase error of 90 degrees corresponding to a 1.0 db gain loss. The explanation for this lies in that only a small part of the aperture is subject to the large phase errors. Nevertheless, the gain expansion must be used with caution.

It is also of interest to note the relative gain reductions of the displacements termed ALAD, ALLD and BPLLD. For a typical f/D of 0.4 the relative sensitivity of these displacements is respectively: 0.051, 0.78 and 0.0033. The tight tolerance required of the axial loss lateral displacement is not surprising as we are off the beam peak. This may not be required in systems where boresight correction is permitted. Similarly the beam peak loss due to lateral displacement may be an insufficient criteria as other factors such as coma lobe pattern distortions[4] may be the determining specification. In all cases where stringent pattern specifications exist comprehensive radiation pattern calculations are necessary using the phase error results of Table I.

When several displacements exist simultaneously, which is usually the case, the resultant phase error may be obtained by adding arithmetically the corresponding errors in Table I. This provides an effective means of error compensation when errors of the same type are involved. For example, an outward axial movement of the Cassegrain subreflector may be compensated by a similar but greater movement outward of the primary feed. Additionally a lateral movement or subreflector droop may be compensated by a subreflector rotation. These compensations are never complete. Although the resultant phase errors are given as combinations of the values in Table I, the resultant gain reduction and radiation pattern distortions must be obtained generally by computer calculations when the phase errors are significant.

Although the phase errors of various displacements are additive, the gain reductions are not if they are of the same type as they may add constructively or compensate. However, gain reductions of errors of different type — axial, lateral and random — are additive as may be seen by substitution in (3).

IV. Illustrative Example

Rusch, Slobin and Sterlzried[8] investigated a nutating Cassegrain subreflector. Their dish diameter was 60 inches and the focal length 25.6". The Cassegrain magnification was 7.08 and "c-a" was 3.17 inches. A HPBW of about 10 minutes of arc was obtained at 90 Gc. The subreflector was nutated through a half angle of 2.06° about a point 0.6 inches behind the subreflector. It is required to find the displaced beam position and the loss of gain.

The subreflector nutation is equivalent to a rotation about its vertex of 2.06° and a displacement of:

$$0.6'' \frac{(2.06)}{57.3} = 0.0216''$$

Combining the two types of displacements from Table II with opposite sign we have:

$$\Delta\alpha \frac{(c-a)}{f}\left[BOf + BOF\right] - \left[\frac{\Delta X}{f} BOf - \frac{\Delta X}{F} BOF\right] =$$

$$\frac{2.06(3.17)}{57.3\,(25.6)}\left[0.83 + 1.0\right] - \left[\frac{0.0216}{25.6}(0.83) - \frac{0.0216}{7.08(25.6)}\right] =$$

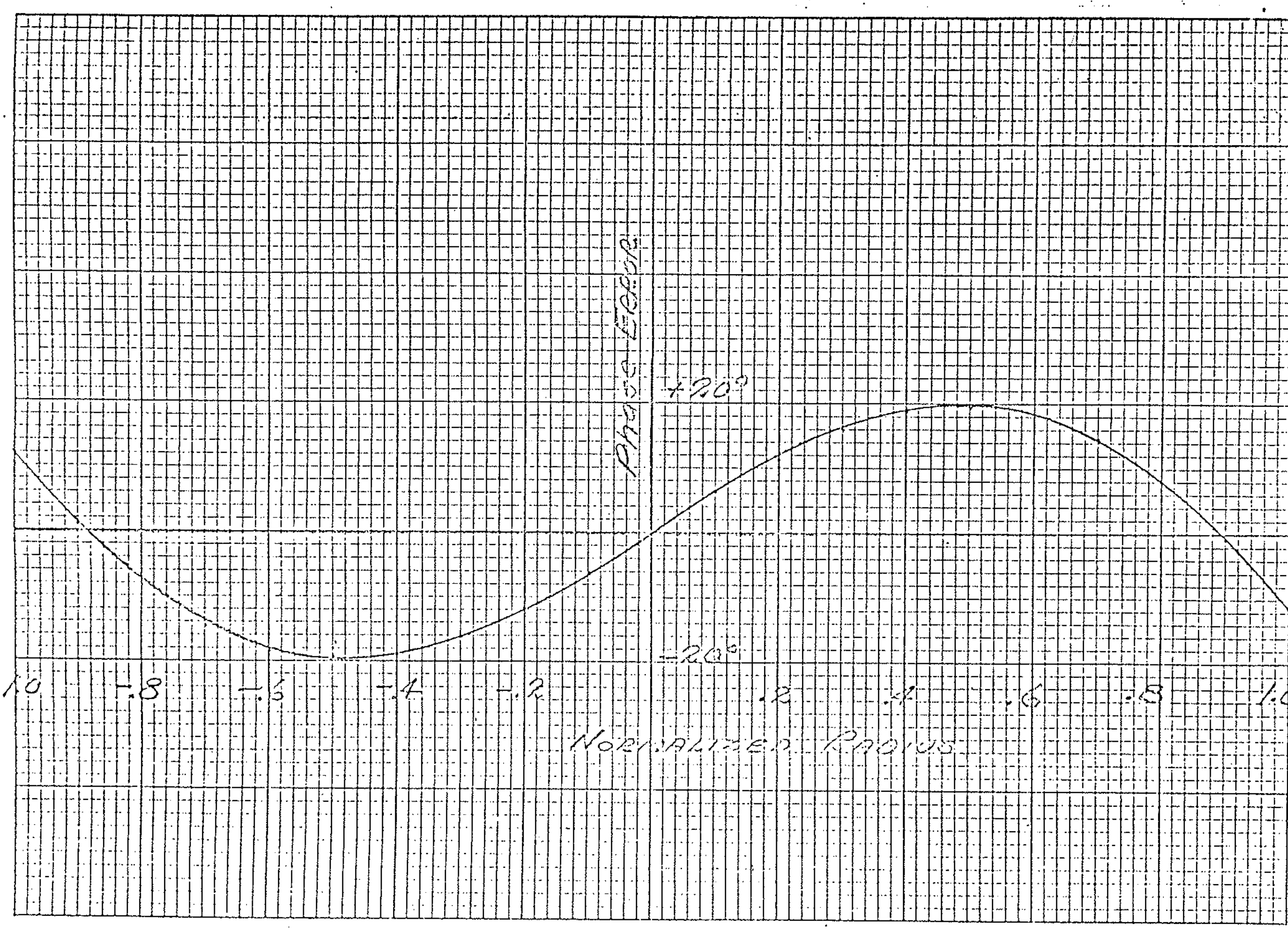
$$0.007565 \text{ radians} = 26 \text{ minutes}$$

The measured beam peak displacement reported was 27.75 minutes.

Similarly the aperture path length error is obtained from Table I as:

$$\left\{\Delta\alpha(c-a)\left[\sin\theta_p + M\sin\theta_f\right] - \Delta X\left[\sin\theta_p - \sin\theta_f\right]\right\}\cos\varphi$$

Removing the linear phase tilt due to the beam shift of 26 minutes we obtain the phase error shown in Fig. 5. Calculated gain loss by the use of Eq. (3) is 0.15 db. Additional gain loss occurs due to the increased spill-over of the displaced subreflector. No measured gain figure was reported.


Phase Error
+20°
-20°
-1.0
-.8
-.6
-.4
-.2
.2
.4
.6
.8
1.0
Normalized Radius

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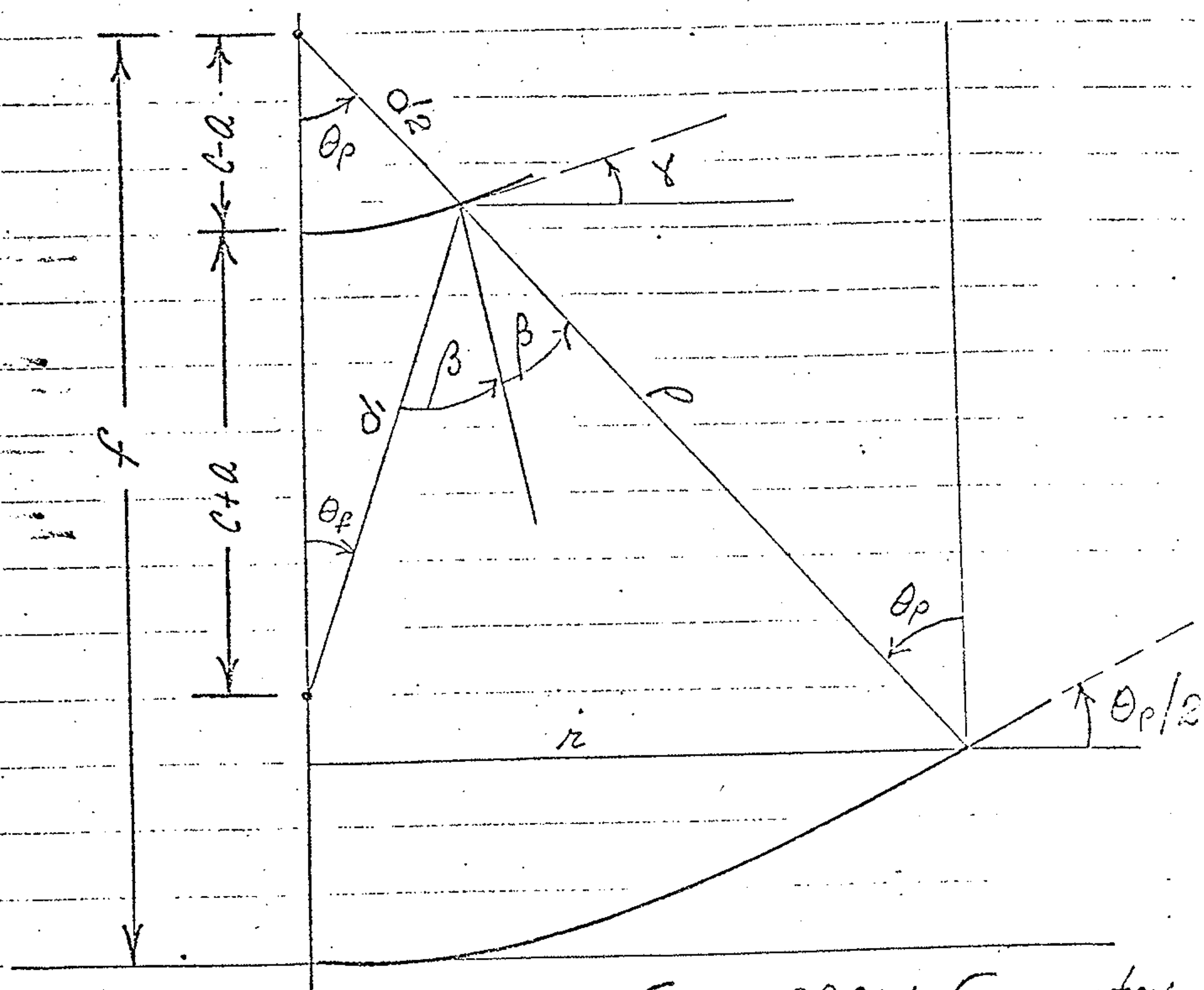
Appendix

1). Notation and Geometry



CASSEGRAIN Geometry.

$$\rho = \frac{2f}{1+\cos\theta_p} = \frac{f}{\cos^2\frac{\theta_p}{2}}$$

$$d_1 = \frac{a(1-e^2)}{1-e\cos\theta_f} \quad ; \quad d_2 = \frac{-a(1-e^2)}{1+e\cos\theta_p}$$

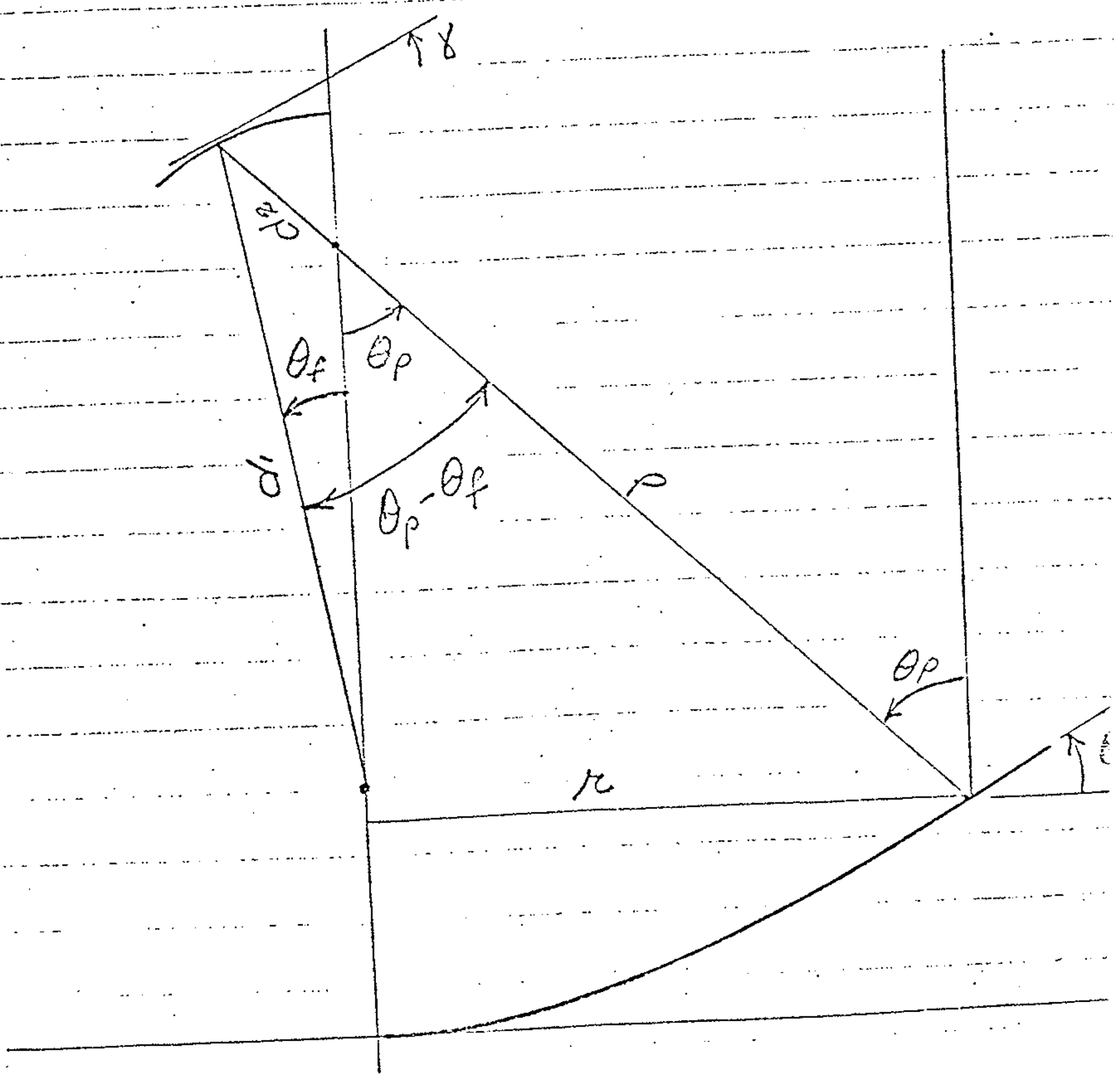
$$e = \frac{c}{a} > 1 \quad ; \quad M = \frac{e+1}{e-1}$$

$F = Mf$

$$\gamma = \frac{\theta_p - \theta_f}{2} \quad ; \quad \beta = \frac{\theta_p + \theta_f}{2}$$

$$\sin\theta_p = \frac{r/f}{1+(r/2f)^2} \quad ; \quad \sin\theta_f = \frac{r/F}{1+(r/2F)^2}$$

GREGORIAN Geometry



$$d_1 = \frac{a(1-e^2)}{1-e\cos\theta_f} \quad ; \quad d_2 = \frac{a(1-e^2)}{1+e\cos\theta_p}$$

$$e = \frac{c}{a} < 1 \quad ; \quad M = \frac{1+e}{1-e}$$

$$F = Mf \quad ; \quad \gamma = \frac{\theta_p + \theta_f}{2}$$

$$\sin\theta_f = \frac{r/f}{1+(r/2f)^2} \quad ; \quad \sin\theta_p = \frac{r/F}{1+(r/2F)^2}$$

2). Sample Derivation

Consider an axial displacement, Δy, of the sub-refl[ector]
The path length error, ϵ, is:

$$\epsilon = 2\Delta y \cos\left(\frac{\theta_p - \theta_f}{2}\right)\cos\left(\frac{\theta_p + \theta_f}{2}\right)$$

$$\epsilon = \Delta y\left[\cos\theta_p + \cos\theta_f\right]$$

normalizing at the aperture center

$$\epsilon = \Delta y\left[(1-\cos\theta_p) + (1-\cos\theta_f)\right]$$

or in terms of the aperture radial coordinate:

$$\epsilon = 2\Delta y\left[\frac{(r/2f)^2}{1+(r/2f)^2} + \frac{(r/2F)^2}{1+(r/2F)^2}\right]$$